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METHOD OF SOLUTION FREDHOLM'S INTEGRAL EQUATION OF THE FIRST KIND*

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Solution of the Fredholm's integral equation of the first kind is approximated by expansion over the system of iterated right-hand sides. The convergence in the mean of the expansion towards the solution is proved. A class of equations with stable iterated right-hand sides in which the solution is reduced to investigating numerical sequences, is singled out.

In the theory of linear filtration an important part is played by the Fredholm's integral equation of the first kind

$$\int_0^T K(t,\tau)\varphi(\tau)\,d\tau = S(t), \quad 0 \leqslant t \leqslant T \tag{1}$$

the solution $\varphi(t)$ of which determines the weight function sought, of a physically feasible linear filter

$$h(t) = \varphi(T-t)$$

The necessary and sufficient condition of existence of a unique solution of equation (1) with closed symmetric kernel $K(t,\tau)$ and $S(t) \in L_2[0, T]$, is given by the Picard theorem /1/. In practice, it is difficult to obtain the solution of (1). The known methods, apart from the numerical ones, include the method of expanding the function $\varphi(t)$ over some complete system of functions /2/, and the method of consecutive approximations /3/.

The function can be expanded into a Fourier series

$$\varphi(t) = \sum_{k=1}^{\infty} \frac{s_k}{\lambda_k} e_k(t) \tag{2}$$

by virtue of the Hilbert-Schmidt theorem /1,2/, but this requires a solution of the corresponding equation of the second kind

$$\int_0^T K(t,\tau) e_k(\tau)\,d\tau = \lambda_k e_k(t)$$

Investigation of the series (2) yields a new method of solving equation (1) with a closed, symmetric positive definite kernel and $S(t) \in L_2[0, T]$.

Solution in the space L_2. The partial sum of the series (2)

$$\varphi_R(t) = \sum_{k=1}^{R} \varphi_k e_k(t)$$

represents the optimal approximation in the mean to the function $\varphi(t) \in L_2[0, T]$. Let us use, instead of the coefficients $\varphi_k = s_k / \lambda_k$, the approximations $\varphi_{kN} = s_k / \lambda_{kN}$. If we approximate λ_{kN}^{-1} by means of a partial sum of the power series which has the form

$$\frac{1}{\lambda_k} = \sum_{i=0}^{\infty} \frac{(\lambda_k - 1)^i}{i!} \left[\frac{d^i}{d\lambda_k^i} \left(\frac{1}{\lambda_k} \right) \Big|_{\lambda_k=1} \right] = \sum_{i=0}^{\infty} (1-\lambda_k)^i$$

near the point $\lambda_k = 1$ and converges uniformly in the region $0 < \lambda_k < 2$, then we have

$$\varphi_{kN} = s_k \sum_{i=0}^{N} (1-\lambda_k)^i \tag{3}$$

From the bilinear expansion of the kernel

$$K(t,\tau) = \sum_{k=1}^{\infty} \lambda_k e_k(t) e_k(\tau)$$

it follows that the square of its norm is

$$B^2 = \int_0^T \int_0^T K^2(t,\tau)\,dt\,d\tau = \sum_{k=1}^{\infty} \lambda_k^2$$

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Let $\lambda_k^* = B^{-1}\lambda_k$. Then

$$B^{*2} = \int_0^T\int_0^T \Big(\sum_{k=1}^{\infty} B^{-1}\lambda_k e_k(t)\, e_k(\tau)\Big)^2 dt\, d\tau = \sum_{k=1}^{\infty} \lambda_k^{*2} = 1$$

the normed eigenvalues satisfy the inequality

$$1 \geqslant \lambda_1^* \geqslant \lambda_2^* \geqslant \ldots \geqslant \lambda_k^* \geqslant \ldots \geqslant 0 \qquad (4)$$

and the approximation (3) is correct. Clearly, the solutions of the equations with unnormed and normed kernels are connected by the simple relation

$$\varphi(t) = B\varphi^*(t)$$

so that from now on we shall assume that the inequality (4) holds also for λ_k.

The necessary and sufficient condition of convergence in the mean is established using the Fischer–Riesz theorem: if

$$\lim_{n,\, m\to\infty} \int_a^b [f_n(x) - f_m(x)]^2 dx = 0, \quad f_n(x) \in L_2[a, b]$$

then $f_n(x)$ converges to $f(x)$ in the mean:

$$\lim_{n\to\infty} f_n(x) = f(x), \; f(x) \in L_2[a, b]$$

Theorem. Let $K(t, \tau)$ be a symmetric, positive definite L_2-kernel. $S(t) \in L_2[0, T]$, and let the equation (1) have a unique solution. Let also $\lambda_k \leqslant 1$ and $e_k(t)$ be the eigenvalues and eigenfunctions of the kernel $K(t, \tau)$, and s_k the coefficients of expansion of $S(t)$ over the system of functions $e_k(t)$. Then the partial sum

$$\varphi_{RN}(t) = \sum_{k=1}^{R} s_k \Big(\sum\nolimits_N\Big) e_k(t), \quad \sum\nolimits_N = \sum_{i=0}^{N} (1-\lambda_k)^i \qquad (5)$$

converges in the mean to the solution $\varphi(t)$ of the equation. Let $Q > R$. Then we have

$$I = \int_0^T [\varphi_{RN}(t) - \varphi_{QM}(t)]^2 dt = \sum_{k=1}^{R} s_k^2 \sum\nolimits_N^2 - 2\sum_{k=1}^{R} s_k^2 \sum\nolimits_N \sum\nolimits_M + \sum_{k=1}^{Q} s_k^2 \sum\nolimits_M^2$$

Since the series (3) converges uniformly, we can write, taking into account (4),

$$\sum\nolimits_L = \frac{1}{\lambda_k} - \varepsilon_L, \quad \varepsilon_L > 0, \quad \lim_{L\to\infty} \varepsilon_L = 0, \quad L = N, M$$

Then

$$I = \sum_{k=1}^{R} s_k^2 \Big[\Big(\frac{1}{\lambda_k} - \varepsilon_N\Big)^2 - 2\Big(\frac{1}{\lambda_k} - \varepsilon_N\Big)\Big(\frac{1}{\lambda_k} - \varepsilon_M\Big) + \Big(\frac{1}{\lambda_k} - \varepsilon_M\Big)^2\Big] + \sum_{k=R+1}^{Q} s_k^2 \Big(\frac{1}{\lambda_k} - \varepsilon_M\Big)^2 =$$

$$(\varepsilon_M - \varepsilon_N)^2 \sum_{k=1}^{R} s_k^2 + \sum_{k=R+1}^{Q} s_k^2 \Big(\frac{1}{\lambda_k} - \varepsilon_M\Big)^2, \quad \lim_{R\to\infty} \sum_{k=1}^{R} s_k^2 = \int_0^T S^2(t)\, dt < \infty$$

$$\lim_{R,\, Q,\, M,\, N\to\infty} = \lim_{R,\, Q\to\infty} \sum_{k=R+1}^{Q} \frac{s_k^2}{\lambda_k^2} = \lim_{R\to Q-1,\, Q\to\infty} \sum_{k=R+1}^{Q} \frac{s_k^2}{\lambda_k^2} = \lim_{Q\to\infty} \frac{s_Q^2}{\lambda_Q^2} = \sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k^2} - \lim_{P\to\infty} \sum_{k=1}^{P} \frac{s_k^2}{\lambda_k^2} = 0$$

The conditions of the Fischer–Riesz theorem hold, therefore $\varphi_{RN}(t)$ converges in the mean to some function $U(t) \in L_2[0, T]$. Substitution of $\varphi_{RN}(t)$ reduces the left hand part of the equation to the form

$$I_1 = \int_0^T K(t, \tau)\, \varphi_{RN}(\tau)\, d\tau = \sum_{k=1}^{R} s_k \lambda_k \Big(\sum\nolimits_N\Big) e_k(t) = \sum_{k=1}^{R} s_k \lambda_k \Big(\frac{1}{\lambda_k} - \varepsilon_N\Big) e_k(t), \quad \lim_{R,\, N\to\infty} I_1 = \lim_{R\to\infty} \sum_{k=1}^{R} s_k e_k(t) = S(t)$$

Thus the function $\varphi_{RN}(t)$ converts, in the limit, the equation into an identity, belongs to the space L_2 and, according to the condition of the theorem, is a unique solution of the equation.

Using the function

$$f_{Rn}(t) = \sum_{k=1}^{R} \lambda_k^n s_k e_k(t), \quad n = 0, 1, \ldots \qquad (6)$$

we can write the partial sum (5) in the form

$$\varphi_{RN}(t) = \sum_{i=0}^{N} (-1)^i \binom{N+1}{i+1} f_{Ri}(t), \quad \binom{p}{n} = \frac{p!}{n!\,(p-n)!} \qquad (7)$$

and the sum (7) converges, by virtue of the absolute convergence of the Fourier series, absolutely to $\varphi(t)$ (in the mean). When $R \to \infty$, the functions (6) converge to

$$f_0(t) = \sum_{k=1}^{\infty} s_k e_k(t) = S(t), \quad f_1(t) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \lambda_k s_i e_k(t) \int_0^T e_k(\tau) e_i(\tau)\, d\tau = \int_0^T \sum_{k=1}^{\infty} \lambda_k e_k(t) e_k(\tau) \times \sum_{i=1}^{\infty} s_i e_i(\tau)\, d\tau = \int_0^T K(t,\tau) S(\tau)\, d\tau$$

$$f_n(t) = \int_0^T \sum_{k=1}^{\infty} \lambda_k e_k(t) e_k(\tau) \sum_{i=1}^{\infty} \lambda_i^{n-1} s_i e_i(\tau)\, d\tau = \int_0^T K(t,\tau) f_{n-1}(\tau)\, d\tau, \quad n = 1, 2, \ldots$$

The iterated right-hand parts of the equation (1), i.e. the functions $f_n(t)$, yield its approximate solution in the mean

$$\varphi_N(t) = \sum_{i=0}^{N} (-1)^i \binom{N+1}{i+1} f_i(t) \tag{8}$$

Example 1. A linear frequency converter with the weight function $h(t)$ is described by the equation

$$\int_{-T}^{T} \cos\omega_1(t-\tau)\, h(\tau)\, d\tau = \cos\omega_2 t$$

The iterated right-hand parts are

$$f_0(t) = \cos\omega_2 t, \; f_n(t) = \alpha\beta^{n-1}\cos\omega_1 t, \; n = 1, 2, \ldots, \quad \alpha = \frac{1}{\omega_1 - \omega_2}\sin(\omega_1 - \omega_2)T + \frac{1}{\omega_1 + \omega_2}\sin(\omega_1 + \omega_2)T, \quad \beta = T + \frac{1}{2\omega_1}\sin 2\omega_1 T$$

The function

$$\varphi_N(t) = (N+1)\cos\omega_2 t + \alpha\sum_{i=1}^{N} (-1)^i \binom{N+1}{i+1} \beta^{i-1}\cos\omega_1 t$$

has no limit as $N \to \infty$, therefore the equation has no solutions and no linear frequency converters of monochromatic oscillations of finite energy exist in the space $L_2[-T, T]$.

Equation with stable, iterated right-hand parts. Solution (8) requires that the iterated right-hand parts of $f_n(t)$ be determined, and in this sense it is not preferable to the method of consecutive approximations /3/. We can however single out a class of equations for which an exact solution can be obtained.

Example 2. The Wiener – Hopf equation of linear filtration /5/ for the angle of heel of a ship /6/ is

$$\int_0^T K(t-\tau)\varphi(\tau)\, d\tau = K(t) - \varphi(t), \quad K(t) = e^{-\alpha t}\left(\cos 2\pi f t + \frac{\alpha}{2\pi f}\sin 2\pi f t\right)$$

The iterated right-hand parts have general form

$$f_n(t) = e^{-\alpha t}(a_n\cos 2\pi f t + b_n \sin 2\pi f t), \; n = 0, 1, 2, \ldots, \quad a_0 = \frac{1}{2}, \quad b_0 = \frac{\alpha}{4\pi f}, \quad a_1 = \frac{T}{4}\left(1 - \frac{\alpha^2}{4\pi^2 f^2}\right), \quad b_1 = \frac{\alpha T}{4\pi f}, \ldots$$

Clearly, the analytic solution of the equation can be written in the analogous form

$$\varphi(t) = e^{-\alpha t}(A\cos 2\pi f t + B\sin 2\pi f t)$$

The unknown A and B are defined from

$$A\int_0^T K(t-\tau)\cos 2\pi f\tau\, d\tau = 1 - A, \quad B\int_0^T K(t-\tau)\sin 2\pi f\tau\, d\tau = \frac{\alpha}{2\pi f} - B \tag{9}$$

For $T = k/f \; (k = 1, 2, \ldots)$, the solution of the system becomes

$$A = 2f[4\pi^2 f^2(k+2f) - k\alpha^2]/\Delta, \; B = 8\pi\alpha f^3/\Delta, \quad \Delta = 4\pi^2 f^2(k+2f)^2 - k^2\alpha^2$$

Example 3. The iterated right-hand parts of the equation are

$$\int_{-\pi/2}^{\pi/2} \cos^2(t-\tau)\varphi(\tau)\, d\tau = \cos^2 t, \quad f_n(t) = \gamma_{0n} + \gamma_{1n}\cos^2 t, \; n = 1, 2, \ldots, \quad \gamma_{0n} = \frac{8}{9\pi}\gamma_{0(n-1)} + \frac{2}{3^{n+1}\pi}, \quad \gamma_{1n} = \frac{4}{3^{n+1}\pi}$$

Since

$$\lim_{N\to\infty} \sum_{i=0}^{N} (-1)^i \binom{N+1}{i+1} \gamma_{mi} = \begin{cases} -1/\pi, & m = 0 \\ 4/\pi, & m = 1 \end{cases}$$

the solution of the equation is

$$\varphi(t) = \frac{4}{\pi}\left(-\frac{1}{4} + \cos^2 t\right)$$

In the examples discussed above the iterated right-hand parts are stable, i.e. they have the form

$$f_n(t) = \sum_{m=0}^{M} \gamma_{mn} g_m(t)$$

When the iterated right-hand parts are stable, the solution (8) can be reduced to the problem of solving systems of the type (9), or in other words, to the study of the limits of numerical sequences

$$\lim_{N\to\infty} \sum_{i=0}^{N} (-1)^i \binom{N+1}{i+1} \gamma_i \tag{10}$$

The stability of the iterated right-hand parts and the existence of the finite limits (10) or of solutions of the systems (9), form the necessary conditions for obtaining an exact solution (8).

Expanding the functions $f_n(t)$ into a power series, we can write the solution of the equation also in the form of a power series

$$\varphi_p(t) = \sum_{p=0}^{\infty} f_p t^p, \quad f_p = \sum_{i=0}^{\infty} (-1)^i \binom{N+1}{i+1} f_{pi}, \quad f_{pi} = \frac{1}{p!} f_i^p(t) = \frac{1}{p!} \int_0^T \frac{\partial^p}{\partial t^p} K(t,\tau) f_{i-1}(\tau)\, d\tau, \quad t = 0$$

converging in the mean to $\varphi(t)$.

Equation with a degenerated kernel. Let the kernel of (1) be degenerate

$$K(t,\tau) = \sum_{j=0}^{m} a_j(t) b_j(\tau)$$

Then the equation can be written in the form

$$\sum_{j=0}^{m} \beta_j a_j(t) = S(t), \quad \beta_j = \int_0^T b_j(\tau)\varphi(\tau)\, d\tau \tag{11}$$

Equation (11) means that if the equation (1) has a solution, then its right-hand part can be written as an expansion in terms of the functions $a_1(t), a_2(t), \ldots, a_p(t), p \leqslant m$. The iterated right-hand parts of the equation

$$f_n(t) = \sum_{j=0}^{m} \alpha_{nj} a_j(t), \quad n = 1, 2, \ldots, \quad \alpha_{ni} = \sum_{r=0}^{m} \alpha_{jr}\alpha_{(n-1)r}, \quad \alpha_{jr} = \int_0^T b_j(t) a_r(t)\, dt$$

are stable, therefore its solution is

$$\varphi(t) = \lim_{N\to\infty} \sum_{j=0}^{m} \sum_{i=0}^{N} (-1)^i \binom{N+1}{i+1} \alpha_{ij} a_j(t) = \sum_{j=0}^{m} \nu_j a_j(t), \quad \sum_{j=0}^{m} \alpha_{0j} a_j(t) = S(t) \tag{12}$$

Substituting (12) into (1), reduces it to the form

$$\int_0^T K(t,\tau)\varphi(\tau)\, d\tau = \sum_{j=0}^{m} \sum_{r=0}^{m} \alpha_{jr}\nu_r a_j(t) = \sum_{j=0}^{m} \beta_j a_j(t) = S(t)$$

which describes a system of linear equations

$$\sum_{r=0}^{m} \alpha_{jr}\nu_r = \beta_j, \quad j = 0, \ldots, m$$

in terms of the unknowns

$$\nu_r = \sum_{i=0}^{\infty} (-1)^i \binom{N+1}{i+1} \alpha_{ir}$$

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